

Rosso-Yamane Theorem on PBW basis of $U_q(A_N)^*$

Yuqun Chen, Hongshan Shao

School of Mathematical Sciences
South China Normal University
Guangzhou 510631, P. R. China
yqchen@scnu.edu.cn
shaohongshan118@163.com

K. P. Shum
Department of Mathematics
The University of Hong Kong
Pokfulam Road, Hong Kong, China (SAR)
kpshum@maths.hku.hk

Abstract: Let $U_q(A_N)$ be the Drinfeld-Jimbo quantum group of type A_N . In this paper, by using Gröbner-Shirshov bases, we give a simple (but not short) proof of the Rosso-Yamane Theorem on PBW basis of $U_q(A_N)$.

Key words: Quantum group; Quantum enveloping algebra; Gröbner-Shirshov basis.

AMS 2000 Subject Classification: 20G42, 16S15, 13P10

1 Introduction

Since any algebra (commutative, associative, Lie), as well as any module over an algebra, can be presented by generators and defining relations, it is important to have a general method to deal with these presentations. Such a method now exists and is called the Gröbner bases method (due to B. Buchberger [18], [19]), or standand bases method (due to H. Hironaka [21]), or Gröbner-Shirshov bases method (due to A. I. Shirshov [35]). The original Shirshov's paper [35] is for Lie algebra presentations, but it can be easily adopted for associative algebra presentations as well, see L. A. Bokut [3] and G. Bergman [1].

Let, for example, $L = \text{Lie}(X | [x_i x_j] - \sum \alpha_{ij}^k x_k, i > j, x_i, x_j, x_k \in X)$ be a Lie algebra over a field (or a commutative ring) k presented by a k -basis X and the multiplication table. Then $S = \{[x_i x_j] - \sum \alpha_{ij}^k x_k | i > j, x_i, x_j, x_k \in X\}$ is a Gröbner-Shirshov basis (subset) of the free Lie algebra $\text{Lie}(X)$ over k . On the other hand, the universal enveloping

*Supported by the NNSF of China (No.10771077) and the NSF of Guangdong Province (No.06025062).

algebra $U(L) = k\langle X | x_i x_j - x_j x_i - \sum \alpha_{ij}^k x_k, i > j, x_i, x_j, x_k \in X \rangle$ is the associative algebra presented by the same set X and the defining relations $S^{(-)}$ (we rewrite S using $[xy] = xy - yx$). There is a general but not difficult result that for any $S \subset Lie(X)$, S is a Gröbner-Shirshov basis in the sense of Lie algebras if and only if $S^{(-)} \subset k\langle X \rangle$ is a Gröbner-Shirshov basis in the sense of associative algebras (see, for example, [9] and [7]). This means that in our case, $S^{(-)}$ is a Gröbner-Shirshov basis (subset) in $k\langle X \rangle$. By Composition-Diamond lemma (see below), the S -irreducible words on X , $Irr(S) = \{x_{i_1} \dots x_{i_k}, i_1 \leq \dots \leq i_k, k \geq 0\}$ form a k -basis of $U(L)$. This is a conceptional proof of the PBW-Theorem by using Gröbner-Shirshov bases theory.

There are many results on Gröbner-Shirshov bases for associative and Lie algebras, as well as for semigroups, groups, conformal algebras, dialgebras, and so on, see, for example, surveys [14], [15], [25] and [8]. Let us mention those for simple Lie algebras and Lie superalgebras via Serre's presentations ([10], [11], [12], [13], [9]), for modules over simple Lie algebras and Iwahori-Hecke algebras ([23], [24], [25]), for Kac-Moody algebras of types $A_n^{(1)}$, $B_n^{(1)}$, $C_n^{(1)}$, $D_n^{(1)}$ ([31], [32], [33]), for Coxeter groups ([17]), for braid groups via Artin-Burau, Artin-Garside and Briman-Ko-Lee presentations ([4], [5] and [6]).

Drinfeld-Jimbo ([20], [22]) presentations for quantized enveloping algebras $U_q(g)$, where g is a semisimple Lie algebra, are a natural source of associative presentations. M. Rosso [34] and I. Yamane [36] found the PBW-basis of $U_q(A_N)$. G. Lusztig [29] and [30], and M. Kashiwara [26] and [27] found the bases of $U_q(g)$ for any semisimple algebra g , as well as for their representations. Their approach work equally well for quantized enveloping algebras associated with arbitrary symmetrizable Cartan matrix, not just those corresponding to finite dimensional Lie algebras. V. K. Kharchenko [28] found the approach to linear bases of quantized enveloping algebras via the so called character Hopf algebras.

In the paper [16], Gröbner-Shirshov bases approach was applied to study $U_q(g)$ for any symmetrizable Cartan matrix. Using this approach, they got a new proof of the triangular decomposition of $U_q(g)$ (see, for example, Yantzen [37]). For $U_q(A_N)$, it was proved by Bokut and Malcolmson [16] that the Jimbo relations (see [36]) of $U_q^+(A_N)$ constitute a Gröbner-Shirshov basis of $U_q^+(A_N)$ in Jimbo generators $x_{ij}, 1 \leq i, j \leq N + 1$ (see below).

In this paper, we give an elementary proof that Jimbo relations S is a Gröbner-Shirshov basis of $U_q^+(A_N)$. For such a purpose, we just check all possible compositions of polynomials from S and proved that all them can be resolved. Also in §1 in this paper, we are giving necessary definitions and Composition-Diamond lemma following Shirshov [35].

2 Preliminaries

We first cite some concepts and results from the literature which are related to the Gröbner-Shirshov bases for associative algebras.

Let k be a field, $k\langle X \rangle$ the free associative algebra over k generated by X and X^* the free monoid generated by X , where the empty word is the identity which is denoted by 1. For a word $w \in X^*$, we denote the length of w by $deg(w)$. Let X^* be a well ordered set. Let $f \in k\langle X \rangle$ with the leading word \bar{f} . Then we call f monic if \bar{f} has coefficient 1.

Definition 2.1 ([35], see also [2], [3]) *Let f and g be two monic polynomials in $k\langle X \rangle$ and $<$ a well order on X^* . Then, there are two kinds of compositions:*

(i) If w is a word such that $w = \bar{f}b = a\bar{g}$ for some $a, b \in X^*$ with $\deg(\bar{f}) + \deg(\bar{g}) > \deg(w)$, then the polynomial $(f, g)_w = fb - ag$ is called the intersection composition of f and g with respect to w .

(ii) If $w = \bar{f} = a\bar{g}b$ for some $a, b \in X^*$, then the polynomial $(f, g)_w = f - agb$ is called the inclusion composition of f and g with respect to w .

Definition 2.2 ([2], [3], cf. [35]) Let $S \subset k\langle X \rangle$ such that every $s \in S$ is monic. Then the composition $(f, g)_w$ is called *trivial modulo (S, w)* if $(f, g)_w = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$ and $\overline{a_i s_i b_i} < w$. If this is the case, then we write

$$(f, g)_w \equiv 0 \pmod{(S, w)}.$$

In general, for $p, q \in k\langle X \rangle$, we write $p \equiv q \pmod{(S, w)}$ which means that $p - q = \sum \alpha_i a_i s_i b_i$, where each $\alpha_i \in k$, $a_i, b_i \in X^*$, $s_i \in S$ and $\overline{a_i s_i b_i} < w$.

Definition 2.3 ([2], [3], cf. [35]) We call the set S with respect to the well order $<$ a *Gröbner-Shirshov set (basis)* in $k\langle X \rangle$ if any composition of polynomials in S is trivial modulo S .

If a subset S of $k\langle X \rangle$ is not a Gröbner-Shirshov basis, then we can add to S all nontrivial compositions of polynomials of S , and by continuing this process (maybe infinitely) many times, we eventually obtain a Gröbner-Shirshov basis $S^{comp} = S^c$. Such a process is called the Shirshov algorithm.

A well order $>$ on X^* is called monomial if it is compatible with the multiplication of words, that is, for $u, v \in X^*$, we have

$$u > v \Rightarrow w_1 u w_2 > w_1 v w_2, \text{ for all } w_1, w_2 \in X^*.$$

A standard example of monomial order on X^* is the deg-lex order to compare two words first by degree and then lexicographically, where X is a linearly ordered set.

The following lemma was first proved by Shirshov [35] for free Lie algebras (with deg-lex order) in 1962 (see also Bokut [2]). In 1976, Bokut [3] specialized the approach of Shirshov to associative algebras (see also Bergman [1]). For the case of commutative polynomials, this lemma is known as the Buchberger's Theorem in [18] and [19].

Lemma 2.4 (Composition-Diamond Lemma) Let k be a field, $A = k\langle X | S \rangle = k\langle X \rangle / Id(S)$ and $<$ a monomial order on X^* , where $Id(S)$ is the ideal of $k\langle X \rangle$ generated by S . Then the following statements are equivalent:

- (i) S is a Gröbner-Shirshov basis.
- (ii) $f \in Id(S) \Rightarrow \bar{f} = a\bar{s}b$ for some $s \in S$ and $a, b \in X^*$.
- (iii) $Irr(S) = \{u \in X^* | u \neq a\bar{s}b, s \in S, a, b \in X^*\}$ is a basis of the algebra $A = k\langle X | S \rangle$. \square

3 Rosso-Yamane theorem on PBW basis of $U_q(A_N)$

Let k be a field, $A = (a_{ij})$ an integral symmetrizable $N \times N$ Cartan matrix so that $a_{ii} = 2$, $a_{ij} \leq 0$ ($i \neq j$) and there exists a diagonal matrix D with diagonal entries d_i which are nonzero integers such that the product DA is symmetric. Let q be a nonzero element of k such that $q^{4d_i} \neq 1$ for each i . Then the quantum enveloping algebra is (see [20], [22])

$$U_q(A) = k\langle X \cup H \cup Y | S^+ \cup K \cup T \cup S^- \rangle,$$

where

$$\begin{aligned} X &= \{x_i\}, \\ H &= \{h_i^{\pm 1}\}, \\ Y &= \{y_i\}, \\ S^+ &= \left\{ \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \binom{1-a_{ij}}{\nu} x_i^{1-a_{ij}-\nu} x_j x_i^\nu, \text{ where } i \neq j, t = q^{2d_i} \right\}, \\ S^- &= \left\{ \sum_{\nu=0}^{1-a_{ij}} (-1)^\nu \binom{1-a_{ij}}{\nu} y_i^{1-a_{ij}-\nu} y_j y_i^\nu, \text{ where } i \neq j, t = q^{2d_i} \right\}, \\ K &= \{h_i h_j - h_j h_i, h_i h_i^{-1} - 1, h_i^{-1} h_i - 1, x_j h_i^{\pm 1} - q^{\mp 1} d_i a_{ij} h_i^{\pm 1} x_j, h_i^{\pm 1} y_j - q^{\mp 1} y_j h_i^{\pm 1}\}, \\ T &= \{x_i y_j - y_j x_i - \delta_{ij} \frac{h_i^2 - h_i^{-2}}{q^{2d_i} - q^{-2d_i}}\} \text{ and} \\ \binom{m}{n}_t &= \begin{cases} \prod_{i=1}^n \frac{t^{m-i+1} - t^{i-m-1}}{t^i - t^{-i}} & (\text{for } m > n > 0), \\ 1 & (\text{for } n = 0 \text{ or } m = n). \end{cases} \end{aligned}$$

Let

$$A = A_N = \begin{pmatrix} 2 & -1 & 0 & \cdots & 0 \\ -1 & 2 & -1 & \cdots & 0 \\ 0 & -1 & 2 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & \cdots & 2 \end{pmatrix} \text{ and } q^8 \neq 1.$$

It is reminded that in this case, the diagonal matrix D is identity.

We introduce some new variables defined by Jimbo (see [36]) which generate $U_q(A_N)$:

$$\tilde{X} = \{x_{ij}, 1 \leq i < j \leq N+1\},$$

where

$$x_{ij} = \begin{cases} x_i & j = i+1, \\ qx_{i,j-1}x_{j-1,j} - q^{-1}x_{j-1,j}x_{i,j-1} & j > i+1. \end{cases}$$

We now order the set \tilde{X} in the following way.

$$x_{mn} > x_{ij} \iff (m, n) >_{lex} (i, j).$$

Let us recall from Yamane [36] the following notation:

$$\begin{aligned}
C_1 &= \{((i, j), (m, n)) | i = m < j < n\}, \\
C_2 &= \{((i, j), (m, n)) | i < m < n < j\}, \\
C_3 &= \{((i, j), (m, n)) | i < m < j = n\}, \\
C_4 &= \{((i, j), (m, n)) | i < m < j < n\}, \\
C_5 &= \{((i, j), (m, n)) | i < j = m < n\}, \\
C_6 &= \{((i, j), (m, n)) | i < j < m < n\}.
\end{aligned}$$

Let the set \tilde{S}^+ consist of Jimbo relations:

$$\begin{aligned}
x_{mn}x_{ij} &- q^{-2}x_{ij}x_{mn} && ((i, j), (m, n)) \in C_1 \cup C_3, \\
x_{mn}x_{ij} &- x_{ij}x_{mn} && ((i, j), (m, n)) \in C_2 \cup C_6, \\
x_{mn}x_{ij} &- x_{ij}x_{mn} + (q^2 - q^{-2})x_{in}x_{mj} && ((i, j), (m, n)) \in C_4, \\
x_{mn}x_{ij} &- q^2x_{ij}x_{mn} + qx_{in} && ((i, j), (m, n)) \in C_5.
\end{aligned}$$

It is easily seen that $U_q^+(A_N) = k\langle \tilde{X} | \tilde{S}^+ \rangle$.

The following theorem is from [16].

Theorem 3.1 ([16] Theorem 4.1) *Let the notation be as before. Then, with the deg-lex order on \tilde{X}^* , \tilde{S}^+ is a Gröbner-Shirshov basis for $k\langle \tilde{X} | \tilde{S}^+ \rangle = U_q^+(A_N)$.*

Proof. We will prove that all compositions in \tilde{S}^+ are trivial modulo \tilde{S}^+ . We consider the following cases.

Case 1. $f = x_{mn}x_{ij} - q^{-2}x_{ij}x_{mn}$, $g = x_{ij}x_{kl} - q^{-2}x_{kl}x_{ij}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f, g)_w = -q^{-2}x_{ij}x_{mn}x_{kl} + q^{-2}x_{mn}x_{kl}x_{ij}.$$

There are four subcases to consider.

| | $((i, j), (m, n)) \in C_1$ | $((i, j), (m, n)) \in C_3$ |
|----------------------------|---------------------------------|--|
| $((k, l), (i, j)) \in C_1$ | 1.1. $((k, l), (m, n)) \in C_1$ | 1.3. $((k, l), (m, n)) \in C_4, C_5 \text{ or } C_6$ |
| $((k, l), (i, j)) \in C_3$ | 1.2. $((k, l), (m, n)) \in C_4$ | 1.4. $((k, l), (m, n)) \in C_3$ |

1.1. $((i, j), (m, n)) \in C_1$, $((k, l), (i, j)) \in C_1$ and $((k, l), (m, n)) \in C_1$.

Then, we have

$$\begin{aligned}
(f, g)_w &\equiv -q^{-4}x_{ij}x_{kl}x_{mn} + q^{-4}x_{kl}x_{mn}x_{ij} \\
&\equiv -q^{-6}x_{kl}x_{ij}x_{mn} + q^{-6}x_{kl}x_{ij}x_{mn} \\
&\equiv 0.
\end{aligned}$$

1.2. $((i, j), (m, n)) \in C_1$, $((k, l), (i, j)) \in C_3$ and $((k, l), (m, n)) \in C_4$.

Then, we have $(i, j) = (m, l)$, $((k, n), (i, j)) \in C_2$ and

$$\begin{aligned}
(f, g)_w &\equiv -q^{-2}x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + q^{-2}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\
&\equiv -q^{-4}x_{kl}x_{ij}x_{mn} + q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\
&\quad + q^{-4}x_{kl}x_{ij}x_{mn} - q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\
&\equiv 0.
\end{aligned}$$

1.3. $((i, j), (m, n)) \in C_3$, $((k, l), (i, j)) \in C_1$ and $((k, l), (m, n)) \in C_4$, C_5 or C_6 .

1.3.1. If $((k, l), (m, n)) \in C_4$ ($m < l$), then $(k, n) = (i, j)$, $((i, j), (m, l)) \in C_2$ and

$$\begin{aligned}
(f, g)_w &\equiv -q^{-2}x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + q^{-2}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\
&\equiv -q^{-4}x_{kl}x_{ij}x_{mn} + q^{-2}(q^2 - q^{-2})x_{ij}x_{kn}x_{ml} + q^{-4}x_{kl}x_{ij}x_{mn} \\
&\quad - q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\
&\equiv 0.
\end{aligned}$$

1.3.2. If $((k, l), (m, n)) \in C_5$ ($m = l$), then $(k, n) = (i, j)$ and

$$\begin{aligned}
(f, g)_w &\equiv -q^{-2}x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + q^2(q^2x_{kl}x_{mn} - qx_{kn})x_{ij} \\
&\equiv -x_{ij}x_{kl}x_{mn} + q^{-1}x_{ij}x_{kn} + x_{kl}x_{mn}x_{ij} - q^{-1}x_{kn}x_{ij} \\
&\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}x_{kl}x_{ij}x_{mn} \\
&\equiv 0.
\end{aligned}$$

1.3.3. If $((k, l), (m, n)) \in C_6$ ($m > l$), then

$$\begin{aligned}
(f, g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + q^{-2}x_{kl}x_{mn}x_{ij} \\
&\equiv -q^{-4}x_{kl}x_{ij}x_{mn} + q^{-4}x_{kl}x_{ij}x_{mn} \\
&\equiv 0.
\end{aligned}$$

1.4. $((i, j), (m, n)) \in C_3$, $((k, l), (i, j)) \in C_3$ and $((k, l), (m, n)) \in C_3$.

This case is similar to 1.1.

Case 2. $f = x_{mn}x_{ij} - q^{-2}x_{ij}x_{mn}$, $g = x_{ij}x_{kl} - x_{kl}x_{ij}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f, g)_w = -q^{-2}x_{ij}x_{mn}x_{kl} + x_{mn}x_{kl}x_{ij}.$$

There are also four subcases to consider.

| | $((i, j), (m, n)) \in C_1$ | $((i, j), (m, n)) \in C_3$ |
|----------------------------|---|---------------------------------|
| $((k, l), (i, j)) \in C_2$ | 2.1. $((k, l), (m, n)) \in C_2, C_3$ or C_4 | 2.3. $((k, l), (m, n)) \in C_2$ |
| $((k, l), (i, j)) \in C_6$ | 2.2. $((k, l), (m, n)) \in C_6$ | 2.4. $((k, l), (m, n)) \in C_6$ |

2.1. $((i, j), (m, n)) \in C_1$, $((k, l), (i, j)) \in C_2$ and $((k, l), (m, n)) \in C_2, C_3$ or C_4 .

2.1.1. If $((k, l), (m, n)) \in C_2$ ($n < l$), then

$$\begin{aligned}
(f, g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + x_{kl}x_{mn}x_{ij} \\
&\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}x_{kl}x_{ij}x_{mn} \\
&\equiv 0.
\end{aligned}$$

2.1.2. If $((k, l), (m, n)) \in C_3$ ($n = l$), then

$$\begin{aligned}(f, g)_w &\equiv -q^{-4}x_{ij}x_{kl}x_{mn} + q^{-2}x_{kl}x_{mn}x_{ij} \\ &\equiv -q^{-4}x_{kl}x_{ij}x_{mn} + q^{-4}x_{kl}x_{ij}x_{mn} \\ &\equiv 0.\end{aligned}$$

2.1.3. If $((k, l), (m, n)) \in C_4$ ($n > l$), then $((k, n), (i, j)) \in C_2$, $((i, j), (m, l)) \in C_1$ and

$$\begin{aligned}(f, g)_w &\equiv -q^{-2}x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + [x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\ &\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} + q^{-2}x_{kl}x_{ij}x_{mn} \\ &\quad - q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\ &\equiv 0.\end{aligned}$$

For the cases 2.2, 2.3 and 2.4, the proofs are similar to 2.1.1.

Case 3. $f = x_{mn}x_{ij} - q^{-2}x_{ij}x_{mn}$, $g = x_{ij}x_{kl} - x_{kl}x_{ij} + (q^2 - q^{-2})x_{kj}x_{il}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f, g)_w = -q^{-2}x_{ij}x_{mn}x_{kl} + x_{mn}x_{kl}x_{ij} - (q^2 - q^{-2})x_{mn}x_{kj}x_{il}.$$

There are two subcases to consider.

| | $((i, j), (m, n)) \in C_1$ | $((i, j), (m, n)) \in C_3$ |
|----------------------------|--|---|
| | 3.1. | 3.2. |
| $((k, l), (i, j)) \in C_4$ | $((k, l), (m, n)), ((k, j), (m, n)) \in C_4$ | $((k, l), (m, n)) \in C_4, C_5 \text{ or } C_6$ $((k, j), (m, n)) \in C_3$ |

3.1. $((i, j), (m, n)) \in C_1$, $((k, l), (i, j)) \in C_4$ and $(k, l), (m, n), ((k, j), (m, n)) \in C_4$.

Then, we have $((k, n), (i, j)) \in C_2$, $((i, l), (m, n)) \in C_1$, $((i, l), (m, j)) \in C_1$, $((m, l), (i, j)) \in C_1$ and

$$\begin{aligned}(f, g)_w &\equiv -q^{-2}x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + [x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\ &\quad - (q^2 - q^{-2})[x_{kj}x_{mn} - (q^2 - q^{-2})x_{kn}x_{mj}]x_{il} \\ &\equiv -q^{-2}[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} + q^{-2}x_{kl}x_{ij}x_{mn} \\ &\quad - (q^2 - q^{-2})x_{kn}x_{ml}x_{ij} - q^{-2}(q^2 - q^{-2})x_{kj}x_{il}x_{mn} + q^{-2}(q^2 - q^{-2})^2x_{kn}x_{il}x_{mj} \\ &\equiv q^{-4}(q^2 - q^{-2})x_{kn}x_{ml}x_{ij} - (q^2 - q^{-2})x_{kn}x_{ml}x_{ij} + q^{-2}(q^2 - q^{-2})x_{kn}x_{ml}x_{ij} \\ &\equiv 0.\end{aligned}$$

3.2. $((i, j), (m, n)) \in C_3$, $((k, l), (i, j)) \in C_4$, $(k, l), (m, n) \in C_4, C_5 \text{ or } C_6$ and $((k, j), (m, n)) \in C_3$.

3.2.1. If $((k, l), (m, n)) \in C_4$ ($l < m$) and $((k, j), (m, n)) \in C_3$, then $((k, n), (i, j)) \in C_3$, $((i, j), (m, l)) \in C_2$, $((i, l), (m, n)) \in C_4$ and

$$\begin{aligned}(f, g)_w &\equiv -q^{-2}x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + [x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\ &\quad - q^{-2}(q^2 - q^{-2})x_{kj}x_{mn}x_{il} \\ &\equiv -q^{-2}[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + q^{-4}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} + q^{-2}x_{kl}x_{ij}x_{mn} \\ &\quad - (q^2 - q^{-2})x_{kn}x_{ij}x_{ml} - q^{-2}(q^2 - q^{-2})x_{kj}[x_{il}x_{mn} - (q^2 - q^{-2})x_{in}x_{ml}] \\ &\equiv 0.\end{aligned}$$

3.2.2. If $((k, l), (m, n)) \in C_5$ ($l = m$) and $((k, j), (m, n)) \in C_3$, then $((k, l), (i, j)) \in C_4$, $((k, n), (i, j)) \in C_3$, $((i, l), (m, n)) \in C_5$ and

$$\begin{aligned}
(f, g)_w &\equiv -q^{-2}x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + (q^2x_{kl}x_{mn} - qx_{kn})x_{ij} - q^{-2}(q^2 - q^{-2})x_{kj}x_{mn}x_{il} \\
&\equiv -[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + q^{-3}x_{kn}x_{ij} + x_{kl}x_{ij}x_{mn} - qx_{kn}x_{ij} \\
&\quad - q^{-2}(q^2 - q^{-2})x_{kj}[q^2x_{il}x_{mn} - qx_{in}] \\
&\equiv q^{-3}x_{kn}x_{ij} - qx_{kn}x_{ij} + q^{-1}(q^2 - q^{-2})x_{kn}x_{ij} \\
&\equiv 0.
\end{aligned}$$

3.2.3. If $((k, l), (m, n)) \in C_6$ ($l > m$) and $((k, j), (m, n)) \in C_3$, then $((i, l), (m, n)) \in C_6$ and

$$\begin{aligned}
(f, g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + x_{kl}x_{mn}x_{ij} - q^{-2}(q^2 - q^{-2})x_{kj}x_{mn}x_{il} \\
&\equiv -q^{-2}[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + q^{-2}x_{kl}x_{ij}x_{mn} - q^{-2}(q^2 - q^{-2})x_{kj}x_{il}x_{mn} \\
&\equiv 0.
\end{aligned}$$

Case 4. $f = x_{mn}x_{ij} - q^{-2}x_{ij}x_{mn}$, $g = x_{ij}x_{kl} - q^2x_{kl}x_{ij} + qx_{kj}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f, g)_w = -q^{-2}x_{ij}x_{mn}x_{kl} + q^2x_{mn}x_{kl}x_{ij} - qx_{mn}x_{kj}.$$

There are two subcases to consider.

| | | |
|----------------------------|----------------------------|----------------------------|
| | $((i, j), (m, n)) \in C_1$ | $((i, j), (m, n)) \in C_3$ |
| | 4.1. | 4.2. |
| $((k, l), (i, j)) \in C_5$ | $((k, l), (m, n)) \in C_5$ | $((k, l), (m, n)) \in C_6$ |
| | $((k, j), (m, n)) \in C_4$ | $((k, j), (m, n)) \in C_3$ |

4.1. $((i, j), (m, n)) \in C_1$, $((k, l), (i, j)) \in C_5$, $((k, l), (m, n)) \in C_5$ and $((k, j), (m, n)) \in C_4$.

Then, we have $((k, n), (i, j)) \in C_2$ ($m = i$) and

$$\begin{aligned}
(f, g)_w &\equiv -q^{-2}x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + q^2(q^2x_{kl}x_{mn} - qx_{kn})x_{ij} \\
&\quad - q[x_{kj}x_{mn} - (q^2 - q^{-2})x_{kn}x_{mj}] \\
&\equiv -(q^2x_{kl}x_{ij} - qx_{kj})x_{mn} + q^{-1}x_{kn}x_{ij} + q^2x_{kl}x_{ij}x_{mn} \\
&\quad - q^3x_{kn}x_{ij} - qx_{kj}x_{mn} + q(q^2 - q^{-2})x_{kn}x_{mj} \\
&\equiv 0.
\end{aligned}$$

4.2. $((i, j), (m, n)) \in C_3$, $((k, l), (i, j)) \in C_5$, $((k, l), (m, n)) \in C_6$ and $((k, j), (m, n)) \in C_3$.

Then, we have

$$\begin{aligned}
(f, g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + q^2x_{kl}x_{mn}x_{ij} - q^{-1}x_{kj}x_{mn} \\
&\equiv -q^{-2}(q^2x_{kl}x_{ij} - qx_{kj})x_{mn} + x_{kl}x_{ij}x_{mn} - q^{-1}x_{kj}x_{mn} \\
&\equiv 0.
\end{aligned}$$

Case 5. $f = x_{mn}x_{ij} - x_{ij}x_{mn}$, $g = x_{ij}x_{kl} - q^{-2}x_{kl}x_{ij}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f, g)_w = -x_{ij}x_{mn}x_{kl} + q^{-2}x_{mn}x_{kl}x_{ij}.$$

There are four subcases to consider.

| | $((i, j), (m, n)) \in C_2$ | $((i, j), (m, n)) \in C_6$ |
|----------------------------|---|---------------------------------|
| $((k, l), (i, j)) \in C_1$ | 5.1. $((k, l), (m, n)) \in C_2, C_3, C_4, C_5$ or C_6 | 5.3. $((k, l), (m, n)) \in C_6$ |
| $((k, l), (i, j)) \in C_3$ | 5.2. $((k, l), (m, n)) \in C_2$ | 5.4. $((k, l), (m, n)) \in C_6$ |

5.1. $((i, j), (m, n)) \in C_2$, $((k, l), (i, j)) \in C_1$, and $((k, l), (m, n)) \in C_2, C_3, C_4, C_5$ or C_6 .

5.1.1. If $((k, l), (m, n)) \in C_2$ ($l > n$), then we have $((k, l), (i, j)) \in C_1$ and

$$\begin{aligned} (f, g)_w &\equiv -x_{ij}x_{kl}x_{mn} + q^{-2}x_{kl}x_{mn}x_{ij} \\ &\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}x_{kl}x_{ij}x_{mn} \\ &\equiv 0. \end{aligned}$$

5.1.2. If $((k, l), (m, n)) \in C_3$ ($l = n$), then

$$\begin{aligned} (f, g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + q^{-4}x_{kl}x_{mn}x_{ij} \\ &\equiv -q^{-4}x_{kl}x_{ij}x_{mn} + q^{-4}x_{kl}x_{ij}x_{mn} \\ &\equiv 0. \end{aligned}$$

5.1.3. If $((k, l), (m, n)) \in C_4$ ($m < l < n$), then we have $((k, l), (i, j)) \in C_1$, $((k, n), (i, j)) \in C_1$, $((i, j), (m, l)) \in C_2$ and

$$\begin{aligned} (f, g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + q^{-2}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\ &\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{ij}x_{kn}x_{ml} + q^{-2}x_{kl}x_{mn}x_{ij} - q^{-2}(q^2 - q^{-2})x_{kn}x_{ml}x_{ij} \\ &\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} + q^{-2}x_{kl}x_{ij}x_{mn} \\ &\quad - q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\ &\equiv 0. \end{aligned}$$

5.1.4. If $((k, l), (m, n)) \in C_5$ ($m = l$), then we have $((k, n), (i, j)) \in C_1$ and

$$\begin{aligned} (f, g)_w &\equiv -x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + q^{-2}(q^2x_{kl}x_{mn} - qx_{kn})x_{ij} \\ &\equiv -q^2x_{ij}x_{kl}x_{mn} + qx_{ij}x_{kn} + x_{kl}x_{mn}x_{ij} - q^{-1}x_{kn}x_{ij} \\ &\equiv -x_{kl}x_{ij}x_{mn} + q^{-1}x_{kn}x_{ij} + x_{kl}x_{ij}x_{mn} - q^{-1}x_{kn}x_{ij} \\ &\equiv 0. \end{aligned}$$

5.1.5. If $((k, l), (m, n)) \in C_6$ ($l < m$), the proof is similar to 5.1.1.

For the cases of 5.2, 5.3 and 5.4, the proofs are also similar to 5.1.1.

Case 6. $f = x_{mn}x_{ij} - x_{ij}x_{mn}$, $g = x_{ij}x_{kl} - x_{kl}x_{ij}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f, g)_w = -x_{ij}x_{mn}x_{kl} + x_{mn}x_{kl}x_{ij}.$$

There are four subcases to consider.

| | | |
|----------------------------|---------------------------------|--|
| | $((i, j), (m, n)) \in C_2$ | $((i, j), (m, n)) \in C_6$ |
| $((k, l), (i, j)) \in C_2$ | 6.1. $((k, l), (m, n)) \in C_2$ | 6.3. $((k, l), (m, n)) \in C_2, C_3, C_4, C_5 \text{ or } C_6$ |
| $((k, l), (i, j)) \in C_6$ | 6.2. $((k, l), (m, n)) \in C_6$ | 6.4. $((k, l), (m, n)) \in C_6$ |

6.1. $((i, j), (m, n)) \in C_2$, $((k, l), (m, n)) \in C_2$ and $((k, l), (m, n)) \in C_2$.

Then, we have

$$\begin{aligned} (f, g)_w &\equiv -x_{ij}x_{kl}x_{mn} + x_{kl}x_{mn}x_{ij} \\ &\equiv -x_{kl}x_{ij}x_{mn} + x_{kl}x_{ij}x_{mn} \\ &\equiv 0. \end{aligned}$$

6.2. $((i, j), (m, n)) \in C_2$, $((k, l), (m, n)) \in C_6$ and $((k, l), (m, n)) \in C_6$.

This case is similar to 6.1.

6.3. $((i, j), (m, n)) \in C_6$, $((k, l), (m, n)) \in C_2$ and $((k, l), (m, n)) \in C_2, C_3, C_4, C_5 \text{ or } C_6$.

6.3.1. If $((k, l), (m, n)) \in C_2$ ($l > n$), the proof is similar to 6.1.

6.3.2. If $((k, l), (m, n)) \in C_3$ ($l = n$), then

$$\begin{aligned} (f, g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + q^{-2}x_{kl}x_{mn}x_{ij} \\ &\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}x_{kl}x_{ij}x_{mn} \\ &\equiv 0. \end{aligned}$$

6.3.3. If $((k, l), (m, n)) \in C_4$ ($m < l < n$), then we have $((k, n), (i, j)) \in C_2$, $((i, j), (m, n)) \in C_6$ and

$$\begin{aligned} (f, g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + [x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\ &\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{ij}x_{kn}x_{ml} + x_{kl}x_{mn}x_{ij} - (q^2 - q^{-2})x_{kn}x_{ml}x_{ij} \\ &\equiv -x_{kl}x_{ij}x_{mn} + (q^2 - q^{-2})x_{kn}x_{ij}x_{ml} + x_{kl}x_{ij}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\ &\equiv 0. \end{aligned}$$

6.3.4. If $((k, l), (m, n)) \in C_5$ ($m = l$), then we have $((k, n), (i, j)) \in C_2$ and

$$\begin{aligned} (f, g)_w &\equiv -x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + (q^2x_{kl}x_{mn} - qx_{kn})x_{ij} \\ &\equiv -q^2x_{ij}x_{kl}x_{mn} + qx_{ij}x_{kn} + q^2x_{kl}x_{mn}x_{ij} - qx_{kn}x_{ij} \\ &\equiv -q^2x_{kl}x_{ij}x_{mn} + qx_{kn}x_{ij} + q^2x_{kl}x_{ij}x_{mn} - qx_{kn}x_{ij} \\ &\equiv 0. \end{aligned}$$

6.3.5. If $((k, l), (m, n)) \in C_6$ ($l < m$), the proof is similar to 6.1.

6.4. $((i, j), (m, n)) \in C_6$, $((k, l), (m, n)) \in C_6$ and $((k, l), (m, n)) \in C_6$.

This case is also similar to 6.1.

Case 7. $f = x_{mn}x_{ij} - x_{ij}x_{mn}$, $g = x_{ij}x_{kl} - x_{kl}x_{ij} + (q^2 - q^{-2})x_{kj}x_{il}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f, g)_w = -x_{ij}x_{mn}x_{kl} + x_{mn}x_{kl}x_{ij} - (q^2 - q^{-2})x_{mn}x_{kj}x_{il}.$$

There are two subcases to consider.

| | $((i, j), (m, n)) \in C_2$ | $((i, j), (m, n)) \in C_6$ |
|----------------------------|--|---|
| | 7.1. | 7.2. |
| $((k, l), (i, j)) \in C_4$ | $((k, l), (m, n)) \in C_2, C_3, C_4, C_5$ or C_6 $((k, j), (m, n)) \in C_2$ | $((k, l), (m, n)),$ $((k, j), (m, n)) \in C_6$ |

7.1. $((i, j), (m, n)) \in C_2$, $((k, l), (i, j)) \in C_4$, $((k, l), (m, n)) \in C_2, C_3, C_4, C_5$ or C_6 and $((k, j), (m, n)) \in C_2$.

7.1.1. If $((k, l), (m, n)) \in C_2$ ($n < l$) and $((k, j), (m, n)) \in C_2$, then we have $((i, l), (m, n)) \in C_2$ and

$$\begin{aligned} (f, g)_w &\equiv -x_{ij}x_{kl}x_{mn} + x_{kl}x_{mn}x_{ij} - (q^2 - q^{-2})x_{kj}x_{mn}x_{il} \\ &\equiv -[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + x_{kl}x_{ij}x_{mn} - (q^2 - q^{-2})x_{kj}x_{il}x_{mn} \\ &\equiv 0. \end{aligned}$$

7.1.2. If $((k, l), (m, n)) \in C_3$ ($n = l$) and $((k, j), (m, n)) \in C_2$, then $((i, l), (m, n)) \in C_3$ and

$$\begin{aligned} (f, g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + q^{-2}x_{kl}x_{mn}x_{ij} - (q^2 - q^{-2})x_{kj}x_{mn}x_{il} \\ &\equiv -q^{-2}[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + q^{-2}x_{kl}x_{ij}x_{mn} - q^{-2}(q^2 - q^{-2})x_{kj}x_{il}x_{mn} \\ &\equiv 0. \end{aligned}$$

7.1.3. If $((k, l), (m, n)) \in C_4$ ($m < l < n$) and $((k, j), (m, n)) \in C_2$, then we obtain $((k, l), (i, j)) \in C_4$, $((k, n), (i, j)) \in C_4$, $((i, j), (m, l)) \in C_2$, $((i, l), (m, n)) \in C_4$ and

$$\begin{aligned} (f, g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + [x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\ &\quad - (q^2 - q^{-2})x_{kj}x_{mn}x_{il} \\ &\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{ij}x_{kn}x_{ml} + x_{kl}x_{mn}x_{ij} - (q^2 - q^{-2})x_{kn}x_{ml}x_{ij} \\ &\quad - (q^2 - q^{-2})x_{kj}[x_{il}x_{mn} - (q^2 - q^{-2})x_{in}x_{ml}] \\ &\equiv -[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + (q^2 - q^{-2})[x_{kn}x_{ij} - (q^2 - q^{-2})x_{kj}x_{in}]x_{ml} \\ &\quad + x_{kl}x_{ij}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\ &\equiv 0. \end{aligned}$$

7.1.4. If $((k, l), (m, n)) \in C_5$ ($m = l$) and $((k, j), (m, n)) \in C_2$, then $((k, n), (i, j)) \in C_4$, $((i, l), (m, n)) \in C_5$ and

$$\begin{aligned} (f, g)_w &\equiv -x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + (q^2x_{kl}x_{mn} - qx_{kn})x_{ij} - (q^2 - q^{-2})x_{kj}x_{mn}x_{il} \\ &\equiv -q^2x_{ij}x_{kl}x_{mn} + qx_{ij}x_{kn} + q^2x_{kl}x_{mn}x_{ij} - qx_{kn}x_{ij} \\ &\quad - (q^2 - q^{-2})x_{kj}(q^2x_{il}x_{mn} - qx_{in}) \\ &\equiv -q^2[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + q[x_{kn}x_{ij} - (q^2 - q^{-2})x_{kj}x_{in}] \\ &\quad + q^2x_{kl}x_{ij}x_{mn} - qx_{kn}x_{ij} - q^2(q^2 - q^{-2})x_{kj}x_{il}x_{mn} + q(q^2 - q^{-2})x_{kj}x_{in} \\ &\equiv 0. \end{aligned}$$

7.1.5. If $((k, l), (m, n)) \in C_6$ ($l < m$) and $((k, j), (m, n)) \in C_2$, then $((i, l), (m, n)) \in C_6$. This case is similar to 7.1.1.

7.2. $((i, j), (m, n)) \in C_6$, $((k, l), (i, j)) \in C_4$, $((k, l), (m, n)), ((k, j), (m, n)) \in C_6$.

This case is also similar to 7.1.1.

Case 8. $f = x_{mn}x_{ij} - x_{ij}x_{mn}$, $g = x_{ij}x_{kl} - q^2x_{kj}x_{il} + qx_{kj}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f, g)_w = -x_{ij}x_{mn}x_{kl} + q^2x_{mn}x_{kl}x_{ij} + qx_{mn}x_{kj}.$$

There are two subcases to consider.

| | $((i, j), (m, n)) \in C_2$ | $((i, j), (m, n)) \in C_6$ |
|----------------------------|--|--|
| | 8.1. | 8.2. |
| $((k, l), (i, j)) \in C_5$ | $((k, l), (m, n)) \in C_6$ $((k, j), (m, n)) \in C_2$ | $((k, l), (m, n)), ((k, j), (m, n)) \in C_6$ |

8.1. $((i, j), (m, n)) \in C_2$, $((k, l), (i, j)) \in C_5$, $((k, l), (m, n)) \in C_6$ and $((k, j), (m, n)) \in C_2$. Then, we have

$$\begin{aligned} (f, g)_w &\equiv -x_{ij}x_{kl}x_{mn} + q^2x_{kl}x_{mn}x_{ij} + qx_{kj}x_{mn} \\ &\equiv -(q^2x_{kl}x_{ij} - qx_{kj})x_{mn} + q^2x_{kl}x_{ij}x_{mn} + qx_{kj}x_{mn} \\ &\equiv 0. \end{aligned}$$

8.2. $((i, j), (m, n)) \in C_6$, $((k, l), (i, j)) \in C_5$, $((k, l), (m, n)), ((k, j), (m, n)) \in C_6$.

This case is similar to 8.1.

Case 9. $f = x_{mn}x_{ij} - x_{ij}x_{mn} + (q^2 - q^{-2})x_{in}x_{mj}$, $g = x_{ij}x_{kl} - q^{-2}x_{kl}x_{ij}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f, g)_w = -x_{ij}x_{mn}x_{kl} + (q^2 - q^{-2})x_{in}x_{mj}x_{kl} + q^{-2}x_{mn}x_{kl}x_{ij}.$$

There are two subcases to consider.

| | $((i, j), (m, n)) \in C_4$ |
|----------------------------|--|
| $((k, l), (i, j)) \in C_1$ | 9.1. $((k, l), (m, n)), ((k, l), (m, j)) \in C_4, C_5 \text{ or } C_6$ |
| $((k, l), (i, j)) \in C_3$ | 9.2. $((k, l), (m, n)) \in C_4$ $((k, l), (m, j)) \in C_3$ |

9.1. $((i, j), (m, n)) \in C_4$, $((k, l), (i, j)) \in C_1$ and $((k, l), (m, n)), ((k, l), (m, j)) \in C_4, C_5$ or C_6 .

9.1.1. If $((k, l), (m, n)), ((k, l), (m, j)) \in C_4$ ($l > m$), then we have $((i, j), (k, n)) \in$

C_1 , $((k, n), (m, l)) \in C_2$, $((k, j), (i, n)) \in C_1$, $((k, l), (i, n)) \in C_1$, $((i, j), (m, l)) \in C_2$ and

$$\begin{aligned}
(f, g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + (q^2 - q^{-2})x_{in}[x_{kl}x_{mj} - (q^2 - q^{-2})x_{kj}x_{ml}] \\
&\quad + q^{-2}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\
&\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{ij}x_{kn}x_{ml} + (q^2 - q^{-2})x_{in}x_{kl}x_{mj} \\
&\quad - (q^2 - q^{-2})^2x_{in}x_{kj}x_{ml} + q^{-2}x_{kl}x_{mn}x_{ij} - q^{-2}(q^2 - q^{-2})x_{kn}x_{ml}x_{ij} \\
&\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + (q^2 - q^{-2})x_{ij}x_{kn}x_{ml} + q^{-2}(q^2 - q^{-2})x_{kl}x_{in}x_{mj} \\
&\quad - q^{-2}(q^2 - q^{-2})^2x_{kj}x_{in}x_{ml} + q^{-2}x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] \\
&\quad - q^{-2}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\
&\equiv (q^2 - q^{-2})x_{ij}x_{kn}x_{ml} - q^{-2}(q^2 - q^{-2})^2x_{kj}x_{in}x_{ml} - q^{-4}(q^2 - q^{-2})x_{ij}x_{kn}x_{ml} \\
&\equiv 0.
\end{aligned}$$

9.1.2. If $((k, l), (m, n)), ((k, l), (m, j)) \in C_5$ ($l = m$), then we have $((i, j), (k, n)) \in C_1$, $((k, l), (i, n)), ((k, j), (i, n)) \in C_1$ and

$$\begin{aligned}
(f, g)_w &\equiv -x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + (q^2 - q^{-2})x_{in}(q^2x_{kl}x_{mj} - qx_{kj}) \\
&\quad + q^{-2}(q^2x_{kl}x_{mn} - qx_{kn})x_{ij} \\
&\equiv -q^2x_{ij}x_{kl}x_{mn} + qx_{ij}x_{kn} + q^2(q^2 - q^{-2})x_{in}x_{kl}x_{mj} - q(q^2 - q^{-2})x_{in}x_{kj} \\
&\quad + x_{kl}x_{mn}x_{ij} - q^{-1}x_{kn}x_{ij} \\
&\equiv -x_{kl}x_{ij}x_{mn} + qx_{ij}x_{kn} + (q^2 - q^{-2})x_{kl}x_{in}x_{mj} - q^{-1}(q^2 - q^{-2})x_{kj}x_{in} \\
&\quad + x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] - q^{-3}x_{ij}x_{kn} \\
&\equiv qx_{ij}x_{kn} - qx_{kj}x_{in} + q^{-3}x_{kj}x_{in} - q^{-3}x_{ij}x_{kn} \\
&\equiv 0.
\end{aligned}$$

9.1.3. If $((k, l), (m, n)), ((k, l), (m, j)) \in C_6$ ($l < m$), then we have $((k, l), (i, n)) \in C_1$ and

$$\begin{aligned}
(f, g)_w &\equiv -x_{ij}x_{kl}x_{mn} - (q^2 - q^{-2})x_{in}x_{kl}x_{mj} + q^{-2}x_{kl}x_{mn}x_{ij} \\
&\equiv -q^{-2}x_{kl}x_{ij}x_{mn} - q^{-2}(q^2 - q^{-2})x_{kl}x_{in}x_{mj} + q^{-2}x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] \\
&\equiv 0.
\end{aligned}$$

9.2. $((i, j), (m, n)) \in C_4$, $((k, l), (i, j)) \in C_3$, $((k, l), (m, n)) \in C_4$ and $((k, l), (m, j)) \in C_3$. Then, we have $((k, n), (i, j)) \in C_2$, $((k, l), (i, n)) \in C_4$, $((i, j), (m, l)) \in C_3$ and

$$\begin{aligned}
(f, g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + q^{-2}(q^2 - q^{-2})x_{in}x_{kl}x_{mj} \\
&\quad + q^{-2}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\
&\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{ij}x_{kn}x_{ml} + q^{-2}(q^2 - q^{-2})x_{in}x_{kl}x_{mj} + q^{-2}x_{kl}x_{mn}x_{ij} \\
&\quad - q^{-2}(q^2 - q^{-2})x_{kn}x_{ml}x_{ij} \\
&\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + (q^2 - q^{-2})x_{kn}x_{ij}x_{ml} + q^{-2}(q^2 - q^{-2})[x_{kl}x_{in} \\
&\quad - (q^2 - q^{-2})x_{kn}x_{il}]x_{mj} + q^{-2}x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] \\
&\quad - q^{-4}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\
&\equiv (q^2 - q^{-2})x_{kn}x_{ij}x_{ml} - q^{-2}(q^2 - q^{-2})x_{kn}x_{il}x_{mj} - q^{-4}(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\
&\equiv 0.
\end{aligned}$$

Case 10. $f = x_{mn}x_{ij} - x_{ij}x_{mn} + (q^2 - q^{-2})x_{in}x_{mj}$, $g = x_{ij}x_{kl} - x_{kl}x_{ij}$, $w = x_{mn}x_{ij}x_{kl}$.
In the case, we have

$$(f, g)_w = -x_{ij}x_{mn}x_{kl} + (q^2 - q^{-2})x_{in}x_{mj}x_{kl} + x_{mn}x_{kl}x_{ij}.$$

There are two subcases to consider.

| | |
|----------------------------|--|
| | $((i, j), (m, n)) \in C_4$ |
| $((k, l), (i, j)) \in C_2$ | 10.1. $((k, l), (m, n)) \in C_2, C_3 \text{ or } C_4$ $((k, l), (m, j)) \in C_2$ |
| $((k, l), (i, j)) \in C_6$ | 10.2. $((k, l), (m, n)), ((k, l), (m, j)) \in C_6$ |

10.1. $((i, j), (m, n)) \in C_4$, $((k, l), (i, j)) \in C_2$, $((k, l), (m, n)) \in C_2, C_3 \text{ or } C_4$ and $((k, l), (m, j)) \in C_2$.

10.1.1. If $((k, l), (m, n)) \in C_2$ ($l > n$), then we have $((k, l), (i, n)) \in C_2$ and

$$\begin{aligned} (f, g)_w &\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{in}x_{kl}x_{mj} + x_{kl}x_{mn}x_{ij} \\ &\equiv -x_{kl}x_{ij}x_{mn} + (q^2 - q^{-2})x_{kl}x_{in}x_{mj} + x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] \\ &\equiv 0. \end{aligned}$$

10.1.2. If $((k, l), (m, n)) \in C_3$ ($l = n$), then we have $((k, l), (i, n)) \in C_3$ and

$$\begin{aligned} (f, g)_w &\equiv -q^{-2}x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{in}x_{kl}x_{mj} + q^{-2}x_{kl}x_{mn}x_{ij} \\ &\equiv -q^{-2}x_{kl}x_{ij}x_{mn} + q^{-2}(q^2 - q^{-2})x_{kl}x_{in}x_{mj} + q^{-2}x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] \\ &\equiv 0. \end{aligned}$$

10.1.3. If $((k, l), (m, n)) \in C_4$ ($l < n$), then we have $((k, n), (i, j)) \in C_2$, $((k, l), (i, n)) \in C_4$, $((i, j), (m, l)) \in C_4$ and

$$\begin{aligned} (f, g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + (q^2 - q^{-2})x_{in}x_{kl}x_{mj} \\ &\quad + [x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\ &\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{ij}x_{kn}x_{ml} + (q^2 - q^{-2})x_{in}x_{kl}x_{mj} \\ &\quad + x_{kl}x_{mn}x_{ij} - (q^2 - q^{-2})x_{kn}x_{ml}x_{ij} \\ &\equiv -x_{kl}x_{ij}x_{mn} + (q^2 - q^{-2})x_{kn}x_{ij}x_{ml} + (q^2 - q^{-2})[x_{kl}x_{in} - (q^2 - q^{-2})x_{kn}x_{il}]x_{mj} \\ &\quad + x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] - (q^2 - q^{-2})x_{kn}[x_{ij}x_{ml} - (q^2 - q^{-2})x_{il}x_{mj}] \\ &\equiv 0. \end{aligned}$$

10.2. $((i, j), (m, n)) \in C_4$, $((k, l), (i, j)) \in C_6$, $((k, l), (m, n)), (k, l), (m, j)) \in C_6$.

This case is similar to 10.1.

Case 11. $f = x_{mn}x_{ij} - x_{ij}x_{mn} + (q^2 - q^{-2})x_{in}x_{mj}$, $g = x_{ij}x_{kl} - x_{kl}x_{ij} + (q^2 - q^{-2})x_{kj}x_{il}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f, g)_w = -x_{ij}x_{mn}x_{kl} + (q^2 - q^{-2})x_{in}x_{mj}x_{kl} + x_{mn}x_{kl}x_{ij} - (q^2 - q^{-2})x_{mn}x_{kj}x_{il},$$

with

| | |
|----------------------------|---|
| | $((i, j), (m, n)) \in C_4$ |
| $((k, l), (i, j)) \in C_4$ | $((k, l), (m, n)), ((k, l), (m, j)) \in C_4, C_5 \text{ or } C_6$ |

11.1. If $((k, l), (m, n)), ((k, l), (m, j)) \in C_4$ ($l > m$), then we have $((k, n), (i, j)) \in C_2$, $((k, l), (i, n)) \in C_4$, $((k, j), (i, n)) \in C_4$, $((i, j), (m, l)) \in C_2$, $((i, l), (m, n)) \in C_4$, $((i, l), (m, j)) \in C_4$ and

$$\begin{aligned}
(f, g)_w &\equiv -x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + (q^2 - q^{-2})x_{in}[x_{kl}x_{mj} - (q^2 - q^{-2})x_{kj}x_{ml}] \\
&\quad + [x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} - (q^2 - q^{-2})[x_{kj}x_{mn} - (q^2 - q^{-2})x_{kn}x_{mj}]x_{il} \\
&\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{ij}x_{kn}x_{ml} + (q^2 - q^{-2})x_{in}x_{kl}x_{mj} \\
&\quad - (q^2 - q^{-2})x_{in}x_{kj}x_{ml} + x_{kl}x_{mn}x_{ij} - (q^2 - q^{-2})x_{kn}x_{ml}x_{ij} \\
&\quad - (q^2 - q^{-2})x_{kj}x_{mn}x_{il} + (q^2 - q^{-2})^2x_{kn}x_{mj}x_{il} \\
&\equiv -[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + (q^2 - q^{-2})x_{kn}x_{ij}x_{ml} + (q^2 - q^{-2})[x_{kj}x_{in} \\
&\quad - (q^2 - q^{-2})x_{kn}x_{il}]x_{mj} - (q^2 - q^{-2})[x_{kj}x_{in} - (q^2 - q^{-2})x_{kn}x_{ij}]x_{ml} \\
&\quad + x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] - (q^2 - q^{-2})x_{kn}x_{ij}x_{ml} \\
&\quad - (q^2 - q^{-2})x_{kj}[x_{il}x_{mn} - (q^2 - q^{-2})x_{in}x_{ml}] \\
&\quad + (q^2 - q^{-2})^2x_{kn}[x_{il}x_{mj} - (q^2 - q^{-2})x_{ij}x_{ml}] \\
&\equiv 0.
\end{aligned}$$

11.2. If $((k, l), (m, n)), ((k, l), (m, j)) \in C_5$ ($l = m$), then we have $((k, n), (i, j)) \in C_2$, $((k, l), (i, n)) \in C_4$, $((k, j), (i, n)) \in C_4$, $((i, l), (m, n)) \in C_5$, $((i, l), (m, j)) \in C_5$ and

$$\begin{aligned}
(f, g)_w &\equiv -x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + (q^2 - q^{-2})x_{in}(q^2x_{kl}x_{mj} - qx_{kj}) + (q^2x_{kl}x_{mn} - qx_{kn})x_{ij} \\
&\quad - (q^2 - q^{-2})[x_{kj}x_{mn} - (q^2 - q^{-2})x_{kn}x_{mj}]x_{il} \\
&\equiv -q^2x_{ij}x_{kl}x_{mn} + qx_{ij}x_{kn} + q^2(q^2 - q^{-2})x_{in}x_{kl}x_{mj} - q(q^2 - q^{-2})x_{in}x_{kj} \\
&\quad + q^2x_{kl}x_{mn}x_{ij} - qx_{kn}x_{ij} - (q^2 - q^{-2})x_{kj}x_{mn}x_{il} + (q^2 - q^{-2})^2x_{kn}x_{mj}x_{il} \\
&\equiv -q^2[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + qx_{kn}x_{ij} + q^2(q^2 - q^{-2})[x_{kl}x_{in} \\
&\quad - (q^2 - q^{-2})x_{kn}x_{il}]x_{mj} - q(q^2 - q^{-2})[x_{kj}x_{in} - (q^2 - q^{-2})x_{kn}x_{ij}] \\
&\quad + q^2x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] - qx_{kn}x_{ij} \\
&\quad - (q^2 - q^{-2})x_{kj}[q^2x_{il}x_{mn} - qx_{in}] + (q^2 - q^{-2})^2x_{kn}[q^2x_{il}x_{mj} - qx_{ij}] \\
&\equiv 0.
\end{aligned}$$

11.3. If $((k, l), (m, n)), ((k, l), (m, j)) \in C_6$ ($l < m$), then $((k, j), (m, n)) \in C_4$, $((k, l), (i, n)) \in C_4$, $((i, l), (m, n)), ((i, l), (m, j)) \in C_6$ and

$$\begin{aligned}
(f, g)_w &\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{in}x_{kl}x_{mj} + x_{kl}x_{mn}x_{ij} \\
&\quad - (q^2 - q^{-2})[x_{kj}x_{mn} - (q^2 - q^{-2})x_{kn}x_{mj}]x_{il} \\
&\equiv -[x_{kl}x_{ij} - (q^2 - q^{-2})x_{kj}x_{il}]x_{mn} + (q^2 - q^{-2})[x_{kl}x_{in} - (q^2 - q^{-2})x_{kn}x_{il}]x_{mj} \\
&\quad + x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] - (q^2 - q^{-2})x_{kj}x_{il}x_{mn} + (q^2 - q^{-2})^2x_{kn}x_{il}x_{mj} \\
&\equiv 0.
\end{aligned}$$

Case 12. $f = x_{mn}x_{ij} - x_{ij}x_{mn} + (q^2 - q^{-2})x_{in}x_{mj}$, $g = x_{ij}x_{kl} - q^2x_{kl}x_{ij} + qx_{kj}$, $w = x_{mn}x_{ij}x_{kl}$, with

| | |
|----------------------------|---|
| | $((i, j), (m, n)) \in C_4$ |
| $((k, l), (i, j)) \in C_5$ | $((k, l), (m, n)), ((k, l), (m, j)) \in C_6$ $((k, j), (m, n)) \in C_4$ $((k, l), (i, n)) \in C_5$ |

In the case, we can deduce that

$$\begin{aligned}
(f, g)_w &= -x_{ij}x_{mn}x_{kl} + (q^2 - q^{-2})x_{in}x_{mj}x_{kl} + q^2x_{mn}x_{kl}x_{ij} - qx_{mn}x_{kj} \\
&\equiv -x_{ij}x_{kl}x_{mn} + (q^2 - q^{-2})x_{in}x_{kl}x_{mj} + q^2x_{kj}x_{mn}x_{ij} \\
&\quad - q[x_{kj}x_{mn} - (q^2 - q^{-2})x_{kn}x_{mj}] \\
&\equiv -(q^2x_{kl}x_{ij} - qx_{kj})x_{mn} + (q^2 - q^{-2})(q^2x_{kl}x_{in} - qx_{kn})x_{mj} \\
&\quad + q^2x_{kl}[x_{ij}x_{mn} - (q^2 - q^{-2})x_{in}x_{mj}] - qx_{kj}x_{mn} + q(q^2 - q^{-2})x_{kn}x_{mj} \\
&\equiv -q^2x_{kl}x_{ij}x_{mn} + qx_{kj}x_{mn} + q^2(q^2 - q^{-2})x_{kl}x_{in}x_{mj} - q(q^2 - q^{-2})x_{kn}x_{mj} \\
&\quad + q^2x_{kl}x_{ij}x_{mn} - q^2(q^2 - q^{-2})x_{kl}x_{in}x_{mj} - qx_{kj}x_{mn} + q(q^2 - q^{-2})x_{kn}x_{mj} \\
&\equiv 0.
\end{aligned}$$

Case 13. $f = x_{mn}x_{ij} - q^2x_{ij}x_{mn} + qx_{in}$, $g = x_{ij}x_{kl} - q^{-2}x_{kl}x_{ij}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f, g)_w = -q^2x_{ij}x_{mn}x_{kl} + qx_{in}x_{kl} + q^{-2}x_{mn}x_{kl}x_{ij}.$$

There are two subcases to consider.

| | |
|----------------------------|---|
| | $((i, j), (m, n)) \in C_5$ |
| $((k, l), (i, j)) \in C_1$ | 13.1. $((k, l), (m, n)) \in C_6$ $((k, l), (i, n)) \in C_1$ |
| $((k, l), (i, j)) \in C_3$ | 13.2. $((k, l), (m, n)) \in C_5$ $((k, l), (i, n)) \in C_4$ |

13.1. $((i, j), (m, n)) \in C_5$, $((k, l), (i, j)) \in C_1$, $((k, l), (m, n)) \in C_6$ and $((k, l), (i, n)) \in C_1$. Then, we have

$$\begin{aligned}
(f, g)_w &= -q^2x_{ij}x_{kl}x_{mn} + q^{-1}x_{kl}x_{in} + q^{-2}x_{kl}x_{mn}x_{ij} \\
&\equiv -x_{kl}x_{ij}x_{mn} + q^{-1}x_{kl}x_{in} + q^{-2}x_{kl}(q^2x_{ij}x_{mn} - qx_{in}) \\
&\equiv -x_{kl}x_{ij}x_{mn} + q^{-1}x_{kl}x_{in} + x_{kl}x_{ij}x_{mn} - q^{-1}x_{kl}x_{in} \\
&\equiv 0.
\end{aligned}$$

13.2. $((i, j), (m, n)) \in C_5$, $((k, l), (i, j)) \in C_3$, $((k, l), (m, n)) \in C_5$ and $((k, l), (i, n)) \in C_4$. Then, we have $((k, n), (i, j)) \in C_2$ and

$$\begin{aligned}
(f, g)_w &\equiv -q^2x_{ij}(q^2x_{kl}x_{mn} - qx_{kn}) + q[x_{kl}x_{in} - (q^2 - q^{-2})x_{kn}x_{il}] \\
&\quad - q^{-2}(q^2x_{kl}x_{mn} - qx_{kn})x_{ij} \\
&\equiv -q^4x_{ij}x_{kl}x_{mn} + q^3x_{ij}x_{kn} + qx_{kl}x_{in} - q(q^2 - q^{-2})x_{kn}x_{il} + x_{kl}x_{mn}x_{ij} \\
&\quad - q^{-1}x_{kn}x_{ij} \\
&\equiv -q^2x_{kl}x_{ij}x_{mn} + q^3x_{kn}x_{ij} + qx_{kl}x_{in} - q^3x_{kn}x_{il} \\
&\quad + q^{-1}x_{kn}x_{il} + q^2x_{kl}x_{ij}x_{mn} - qx_{kl}x_{in} - q^{-1}x_{kn}x_{ij} \\
&\equiv 0.
\end{aligned}$$

Case 14. $f = x_{mn}x_{ij} - q^2x_{ij}x_{mn} + qx_{in}$, $g = x_{ij}x_{kl} - x_{kl}x_{ij}$, $w = x_{mn}x_{ij}x_{kl}$.

In the case, we have

$$(f, g)_w = -q^2x_{ij}x_{mn}x_{kl} + qx_{in}x_{kl} + x_{mn}x_{kl}x_{ij}.$$

There are two subcases to consider.

| | |
|----------------------------|--|
| | $((i, j), (m, n)) \in C_5$ |
| $((k, l), (i, j)) \in C_2$ | 14.1. $((k, l), (m, n)), ((k, l), (i, n)) \in C_2, C_3$ or C_4 |
| $((k, l), (i, j)) \in C_6$ | 14.2. $((k, l), (m, n)), ((k, l), (i, n)) \in C_6$ |

14.1. $((i, j), (m, n)) \in C_5$, $((k, l), (i, j)) \in C_2$ and $((k, l), (m, n)), ((k, l), (i, n)) \in C_2, C_3$ or C_4 .

14.1.1. If $((k, l), (m, n))$ and $((k, l), (i, n)) \in C_2$ ($l > n$), then

$$\begin{aligned} (f, g)_w &= -q^2x_{ij}x_{kl}x_{mn} + qx_{kl}x_{in} + x_{kl}x_{mn}x_{ij} \\ &\equiv -q^2x_{kl}x_{ij}x_{mn} + qx_{kl}x_{in} + x_{kl}(q^2x_{ij}x_{mn} - qx_{in}) \\ &\equiv 0. \end{aligned}$$

14.1.2. If $((k, l), (m, n))$ and $((k, l), (i, n)) \in C_3$ ($l = n$), then

$$\begin{aligned} (f, g)_w &= -x_{ij}x_{kl}x_{mn} + q^{-1}x_{kl}x_{in} + q^{-2}x_{kl}x_{mn}x_{ij} \\ &\equiv -x_{kl}x_{ij}x_{mn} + q^{-1}x_{kl}x_{in} + x_{kl}x_{ij}x_{mn} - q^{-1}x_{kl}x_{in} \\ &\equiv 0. \end{aligned}$$

14.1.3. If $((k, l), (m, n)), ((k, l), (i, n)) \in C_4$ ($l < n$), then we have $((k, n), (i, j)) \in C_2$, $((i, j), (m, l)) \in C_5$ and

$$\begin{aligned} (f, g)_w &\equiv -q^2x_{ij}[x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}] + q[x_{kl}x_{in} - (q^2 - q^{-2})x_{kn}x_{il}] \\ &\quad + [x_{kl}x_{mn} - (q^2 - q^{-2})x_{kn}x_{ml}]x_{ij} \\ &\equiv -q^2x_{ij}x_{kl}x_{mn} + q^2(q^2 - q^{-2})x_{ij}x_{kn}x_{ml} + qx_{kl}x_{in} - q(q^2 - q^{-2})x_{kn}x_{il} \\ &\quad + x_{kl}x_{mn}x_{ij} - (q^2 - q^{-2})x_{kn}x_{ml}x_{ij} \\ &\equiv -q^2x_{kl}x_{ij}x_{mn} + q^2(q^2 - q^{-2})x_{kn}x_{ij}x_{ml} + qx_{kl}x_{in} - q(q^2 - q^{-2})x_{kn}x_{il} \\ &\quad + x_{kn}(q^2x_{ij}x_{mn} - qx_{in}) - (q^2 - q^{-2})x_{kn}(q^2x_{ij}x_{mn} - qx_{il}) \\ &\equiv 0. \end{aligned}$$

14.2. $((i, j), (m, n)) \in C_5$, $((k, l), (i, j)) \in C_6$ and $((k, l), (m, n)), ((k, l), (i, n)) \in C_6$.

This case is similar to 14.1.1.

Case 15. $f = x_{mn}x_{ij} - q^2x_{ij}x_{mn} + qx_{in}$, $g = x_{ij}x_{kl} - x_{kl}x_{ij} + (q^2 - q^{-2})x_{kj}x_{il}$, $w = x_{mn}x_{ij}x_{kl}$, with

| | |
|----------------------------|--|
| | $((i, j), (m, n)) \in C_5$ |
| $((k, l), (i, j)) \in C_4$ | $((k, l), (m, n)) \in C_6$ $((k, l), (i, n)) \in C_4$ $((k, j), (m, n)) \in C_5$ $((i, l), (m, n)) \in C_6$ |

Then, we have

$$\begin{aligned}
(f, g)_w &= -q^2 x_{ij} x_{mn} x_{kl} + q x_{in} x_{kl} + x_{mn} x_{kl} x_{ij} - (q^2 - q^{-2}) x_{mn} x_{kj} x_{il} \\
&\equiv -q^2 x_{ij} x_{kl} x_{mn} + q [x_{kl} x_{in} - (q^2 - q^{-2}) x_{kn} x_{il}] + x_{kl} x_{mn} x_{ij} \\
&\quad - (q^2 - q^{-2}) (q^2 x_{kj} x_{mn} - q x_{kn}) x_{il} \\
&\equiv -q^2 [x_{kl} x_{ij} - (q^2 - q^{-2}) x_{kj} x_{il}] x_{mn} + q x_{kl} x_{in} - q (q^2 - q^{-2}) x_{kn} x_{il} \\
&\quad + x_{kl} (q^2 x_{ij} x_{mn} - q x_{in}) - q^2 (q^2 - q^{-2}) x_{kj} x_{mn} x_{il} + q (q^2 - q^{-2}) x_{kn} x_{il} \\
&\equiv -q^2 x_{kl} x_{ij} x_{mn} + q^2 (q^2 - q^{-2}) x_{kj} x_{il} x_{mn} + q x_{kl} x_{in} - q (q^2 - q^{-2}) x_{kn} x_{il} \\
&\quad + q^2 x_{kl} x_{ij} x_{mn} - q x_{kl} x_{in} - q^2 (q^2 - q^{-2}) x_{kj} x_{il} x_{mn} + q (q^2 - q^{-2}) x_{kn} x_{il} \\
&\equiv 0.
\end{aligned}$$

Case 16. $f = x_{mn} x_{ij} - q^2 x_{ij} x_{mn} + q x_{in}$, $g = x_{ij} x_{kl} - q^2 x_{kl} x_{ij} + q x_{kj}$, $w = x_{mn} x_{ij} x_{kl}$, with

| | |
|----------------------------|---|
| | $((i, j), (m, n)) \in C_5$ |
| $((k, l), (i, j)) \in C_5$ | $((k, l), (m, n)) \in C_6$ $((k, l), (i, n)), ((k, j), (m, n)) \in C_5$ |

In the case, we have

$$\begin{aligned}
(f, g)_w &= -q^2 x_{ij} x_{mn} x_{kl} + q x_{in} x_{kl} + q^2 x_{mn} x_{kl} x_{ij} - q x_{mn} x_{kj} \\
&\equiv -q^2 x_{ij} x_{kl} x_{mn} + q (q^2 x_{kl} x_{in} - q x_{kn}) + q^2 x_{kl} x_{mn} x_{ij} - q (q^2 x_{kj} x_{mn} - q x_{kn}) \\
&\equiv -q^2 (q^2 x_{kl} x_{ij} - q x_{kj}) x_{mn} + q^3 x_{kl} x_{in} - q^2 x_{kn} + q^2 x_{kl} (q^2 x_{ij} x_{mn} - q x_{in}) \\
&\quad - q^3 x_{kj} x_{mn} + q^2 x_{kn} \\
&\equiv 0.
\end{aligned}$$

Thus, \tilde{S}^+ is a Gröbner-Shirshov basis. This completes the proof of Theorem 3.1. \square

Similarly, with the deg-lex order on \tilde{Y}^* , \tilde{S}^- is a Gröbner-Shirshov basis for $U_q^-(A_N) = k\langle \tilde{Y} | \tilde{S}^- \rangle$.

We now use the same notation as before. Order the generators by: $x_i > x_j$, $h_i > h_i^{-1} > h_j > h_j^{-1}$, $y_i > y_j$ if $i > j$, and $x_i > h_j^{\pm 1} > y_m$ for all i, j, m . Then we obtain a well order (deg-lex) on $\tilde{X} \cup H \cup \tilde{Y}$. Thus, by Theorem 3.1, we re-obtain the following theorem in [16].

Theorem 3.2 ([16] Theorem 2.7) *Let the notation be as before. Then with the deg-lex order on $\{\tilde{X} \cup H \cup \tilde{Y}\}^*$, $\tilde{S}^+ \cup T \cup K \cup \tilde{S}^-$ is a Gröbner-Shirshov basis for $U_q(A_N) = k\langle \tilde{X} \cup H \cup \tilde{Y} | \tilde{S}^+ \cup T \cup K \cup \tilde{S}^- \rangle$.*

Acknowledgement: The authors would like to express their deepest gratitude to Professor L. A. Bokut for his kind guidance, useful discussions and enthusiastic encouragement.

References

- [1] G. M. Bergman, The diamond lemma for ring theory, *Adv. in Math.*, 29, 178-218(1978).
- [2] L. A. Bokut, Unsolvability of the word problem, and subalgebras of finitely presented Lie algebras, *Izv. Akad. Nauk. SSSR Ser. Mat.*, 36, 1173-1219(1972).
- [3] L. A. Bokut, Imbeddings into simple associative algebras, *Algebra i Logika*, 15, 117-142(1976).
- [4] L. A. Bokut, Gröbner-Shirshov bases for the braid groups in the Briman-Ko-Lee generators, submitted.
- [5] L. A. Bokut, Gröbner-Shirshov bases for braid groups in Artin-Garside generators, *J. Symbolic Compu.*, Doi:10.1016/j.jsc.2007.02.003. Available online 17 November 2007.
- [6] L. A. Bokut, V. V. Chainikov and K. P. Shum, Markov and Artin normal form theorem for braid groups, *Comm. Algebra*, 35, 2105-2115(2007).
- [7] L. A. Bokut and Yuqun Chen, Gröbner-Shirshov bases for free Lie algebras: after A. I. Shirshov, *SEA. Bull. Math.*, 31, 1057-1076(2007).
- [8] L. Bokut and Yuqun Chen, Gröbner-Shirshov bases: some new results, Proceedings of the ICAC2007, World Scientific, 2008.
- [9] L. A. Bokut, S. J. Kang, K. H. Lee and P. Malcolmson, Gröbner-Shirshov bases for Lie superalgebras and their universal enveloping algebras, *J. Algebra*, 217(2), 461-495(1999).
- [10] L. A. Bokut and A. A. Klein, Serre relations and Gröbner-Shirshov bases for simple Lie algebras I, *Internat. J. Algebra Comput.*, 6(4), 389-400(1996).
- [11] L. A. Bokut and A. A. Klein, Serre relations and Gröbner-Shirshov bases for simple Lie algebras II, *Internat. J. Algebra Comput.*, 6(4), 401-412(1996).
- [12] L. A. Bokut and A. A. Klein, Gröbner-Shirshov bases for exceptional Lie algebras I, *J. Pure Applied Algebra*, 133, 51-57(1998).
- [13] L. A. Bokut and A. A. Klein, Gröbner-Shirshov bases for exceptional Lie algebras E6, E7, E8. *Algebra and Combinatorics (Hong Kong)*, 37-46, Springer, Singapore, 1999.
- [14] L. A. Bokut and P. Kolesnikov, Gröbner-Shirshov bases: from their incipency to the present, *Journal of Mathematical Sciences*, 116(1), 2894-2916(2003).
- [15] L. A. Bokut and P. Kolesnikov, Gröbner-Shirshov bases, conformal algebras and pseudo-algebras, *Journal of Mathematical Sciences*, 131, 5962-6003(2005).
- [16] L. A. Bokut and P. Malcolmson, Gröbner-Shirshov basis for quantum enveloping algebras, *Israel Journal of Mathematics*, 96, 97-113(1996).
- [17] L. A. Bokut and L.-S. Shiao, Gröbner-Shirshov bases for Coxeter groups, *Comm. Algebra*, 29(9), 4305-4319(2001).

- [18] B. Buchberger, An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal [in German], Ph.D thesis, University of Innsbruck, Austria, 1965.
- [19] B. Buchberger, An algorithmical criteria for the solvability of algebraic systems of equations [in German], *Aequationes Math.*, 4, 374-383(1970).
- [20] V. G. Drinfeld, Hopf algebra and the quantum Yang-Baxter equation, *Doklady Akademii Nauk SSSR*, 283(5), 1060-1064(1985).
- [21] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero I, II. *Ann. of Math.*, 79, 109-203(1964); *ibid.* 79, 205-326(1964).
- [22] M. Jimbo, A q-difference analogue of $U(G)$ and the Yang-Baxter equation, *Letters in Mathematical Physics*, 10(1), 63-69(1985).
- [23] S.-J. Kang and K.-H. Lee, Gröbner-Shirshov bases for representation theory, *J. Korean Math. Soc.*, 37(1), 55-72(2000).
- [24] S.-J. Kang and K.-H. Lee, Gröbner-Shirshov basis theory for irreducible $sl_{(n+1)}$ -modules, *J. Algebra*, 232, 1-20(2000).
- [25] S.-J. Kang, I.-S. Lee, K.-H. Lee and H. OH, Hecke algebras, Speche modules and Gröbner-Shirshov bases, *J. Algebra*, 252, 258-292(2002).
- [26] M. Kashiwara, Crystalizing the q-analogue of universal enveloping algebras, *Commun. Math. Phys.*, 133, 249-260(1990).
- [27] M. Kashiwara, On crystal bases of the q-analogue of universal enveloping algebras, *Duke Math. J.*, 63, 465-516(1991).
- [28] V. K. Kharchenko, A quantum analog of the Poincare-Birkhoff-Witt theorem, *Algebra i Logika*, 38(4), 259-276(1999).
- [29] G. Lusztig, Canonical bases arising from quantized enveloping algebra, *J. Amer. Math. Soc.*, 3(2), 447-498(1990).
- [30] G. Lusztig, Canonical bases arising from quantized enveloping algebra II, *Progr. Theor. Phys. Suppl.*, 102, 175-201(1990).
- [31] E. N. Poroshenko, Gröbner-Shirshov bases for the Kac-Moody algebras of the type $A_n^{(1)}$, *Comm. Algebra*, 30(6), 2617-2637(2002).
- [32] E. N. Poroshenko, Gröbner-Shirshov bases for the Kac-Moody algebras of the type $C_n^{(1)}$ and $D_n^{(1)}$, *Vestn. Novosib. Gos. Univ., Ser. Mat. Mekh. Inform.*, 2(1), 58-70(2002) (in Russian).
- [33] E. N. Poroshenko, Gröbner-Shirshov bases for the Kac-Moody algebras of the type $B_n^{(1)}$, *Int. J. Math. Game Theory Algebra*, 13(2), 117-128(2003).
- [34] M. Rosso, An analogue of the Poincare-Birkhoff-Witt theorem and the universal R-matrix of $U_q(sl(N+1))$, *Comm. Math. Phys.*, 124(2), 307-318(1989).

- [35] A. I. Shirshov, Some algorithmic problem for Lie algebras, *Sibirsk. Mat. Z.*, 3, 292-296(1962) (in Russian); English translation in SIGSAM Bull., 33(2), 3-6(1999).
- [36] I. Yamane, A Poincare-Birkhoff-Witt theorem for quantized universal enveloping algebras of type A_N , *Publ., RIMS. Kyoto Univ.*, 25(3), 503-520(1989).
- [37] J. C. Yantzen, Lectures on quantum groups, Graduate Texts in Mathematics, Vol. 6, AMS, 1996.